

# Massive test particles motion in Kaluza-Klein gravity

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**Abstract** A class of static, vacuum solutions of (free-electromagnetic) Kaluza-Klein equations with three-dimensional spherical symmetry is studied. In order to explore the dynamic in such spacetimes, geodesic equations are obtained and the effective potential for massive test particles is analysed. Particular attention is devoted to the properties of the four-dimensional counterpart of these solutions in their Schwarzschild limit. A modification of the circular stable orbits compared with the Schwarzschild case is investigated.

**Keywords** Kaluza klein · Generalized Schwarzschild solution (GSS) · Circular orbits

## 1 Introduction

Extra dimensional theories are candidate for the great unification, being based on the effort to extend to other fields the geometrical picture of gravitation ([1], [2]). Indeed, some cosmological models, including for instance strings or brane worlds, take an implement of the number of dimensions of the spacetime to the five of the original Kaluza Klein model or more dimensions ([3], [4], [5], [6], [7]). On one side great interest is involved to provide a theoretical model able to explain the role extra dimensions and their compatibility in a world that looks like a four dimensional one. On the other side any experimental observation that could be compatible with such theories could be a

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strong constraint concerning their validity. Moreover, scalar fields which are naturally provided by such theories play a crucial role in the dynamics of the present inflationary models (see for example [8]). 5D Kaluza Klein models provide the geometrization of the electromagnetism and a scalar field associated to extra dimensional component of the metric. The gauge invariance arises as a spacetime symmetry realized imposing the invariance for translations on the compactified fifth dimension. Nevertheless, the study of test particle dynamics shows a great problem of such theory, known as the charge-mass puzzle. It is possible to recover the Lorentz equation for the particle motion, but the charge-mass ratio does not match with any observed particle because the theory provide huge massive modes near the Planck scale. Some works propose a solution making a revision of the approach to the particle dynamics which is usually adopted in these models; for instance in [9], a definition of a 5D particle as a localized matter distribution in the ordinary 4D spacetime but as a delocalized one on the fifth dimension is considered. It leads to a different definition of mass that solves the charge-mass puzzle.

In this work we study test particles motion in a five dimensional, electromagnetic-free, Kaluza-Klein (KK) model. As an extension of the Schwarzschild solution in a 5D scenario, we consider here a vacuum solution of KK equations with 3D-spherical symmetry. Using an effective potential approach to the motion, we are able to find the last circular orbit radius and in particular the last stable circular orbits radius of a charged or neutral test particle. The work is motivated by the aim to provide an experimental constraints on the validity of multidimensional gravity theory exploring the dynamical effects of the extra compactified dimension. The presence of such a dimension should produce a non trivial departure from the dynamics in the corresponding 4D counterparts of these solutions. At first, we analyze the test particles motion by a standard approach to the Kaluza-Klein dynamic, therefore performing a dimensional reduction to four dimensions of a 5D free particle following a 5D geodesic. Then we compare this approach with the new one realized in [9], based on a Papapetrou multipole expansion of a 5D energy-momentum tensor which is supposed to be picked along a 4D-world tube.

The paper is organized as follows: in Sec. 2 we review some fundamental statements of KK model. In Sec. 3 we examine test particles motion reviewing first the geodesic approach and then considering the dynamics from point of view of the multipole expansion. In Sec. 4 we review the circular motion of a test particle in Schwarzschild geometry by mean of the effective potential approach. In Sec. 5 we recall some of the properties of the generalized Schwarzschild solution. Finally, in Sec.6 we explore the dynamics in such spacetimes, we find an effective potential and we study circular orbits, either in the standard scheme of the motion in KK gravity, either in the approach *à la* Papapetrou. The paper will end in Sec.7 where concluding remarks follow.

## 2 Five dimensional Kaluza-Klein Model

The 5D compactified KK model is settled by the following assumptions (see for instance [10],[12],[13]). The 5D-manifold  $\mathcal{M}^5$  is a direct product  $\mathcal{M}^4 \otimes \mathcal{S}^1$ , between the ordinary 4D-spacetime  $\mathcal{M}^4$  and the space-like loop  $\mathcal{S}^1$ . To make the extra dimension

unobservable its size is assumed to be below the present observational bound<sup>1</sup> (Compactification hypothesis). Metrics components do not depend on the fifth coordinate (Cylindricity hypothesis): such a scenario could be realised assuming we are working at the lowest order of the Fourier expansion along the fifth dimension, providing then an effective theory. Finally, we assume that the (55)- component of the metrics is a scalar. Such a setup results in a breaking of the 5D covariance and the 5D Equivalence Principle ([9]); noticeably, only traslations along the fifth dimension are allowed and by this way the abelian gauge invariance of the electromagnetism is realised in KK model as a coordinate transformation in  $S^1$ . According to the KK reduction the 5D line element reads<sup>2</sup> as follows:

$$ds_{(5)}^2 = g_{\mu\nu}dx^\mu dx^\nu - \phi^2 (dx_5 + ekA_\mu dx^\mu)^2. \quad (1)$$

We adopt coordinates  $x^\mu$  for ordinary 4D-spacetime while  $x^5$  is the angle parameter for the fifth circular dimension. The extra scalar field  $\phi$  we have in the model is the scale factor governing the expansion of the extra dimension, being  $\phi^2 = -g_{55}$ ;  $A_\mu$  represents the electromagnetic field and  $g_{\mu\nu}$  is the usual 4D metric tensor;  $ek$  is a dimensional constant such that  $e^2 k^2 = (4G)/c^2$ . In this work we just concern our analysis to those electromagnetic free -solutions ( $A_\mu = 0$ ), i.e. we deal with a pure scalar tensor theory (for a discussion of the role of the scalar field in the KK paradigm see for example [14]-[15]).

### 3 Particle dynamics in Kaluza Klein models

Here we briefly review the geodesic approach to motion in KK model and then we discuss the main features of the Papapetrou revised approach to the motion recently appeared in literature.

#### 3.1 Geodesic approach to motion in KK model

Borrowing the formulation of motion from the 4D theory, a first approach is simply to assume that the particle motion is governed by the Action

$$S_{(5)} \equiv -\mu_{(5)} \int ds_{(5)}, \quad (2)$$

where the mass parameter  $\mu_{(5)}$  is assumed to be constant, according to the assumption of equivalence between the motion of the particle and the 5D geodesic trajectory (see for example [13],[10]). From Action (2) the 5D equations is obtained:

$$\omega^A{}^{(5)} \nabla_A \omega^B = 0. \quad (3)$$

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<sup>1</sup> This means

$$L_{(5)} \equiv \int d^5x \sqrt{g_{55}} < 10^{-18} cm.$$

<sup>2</sup> With latin capital letters  $A$  we label the five-dimensional indices, where they run in  $\{0, 1, 2, 3, 5\}$ , Greek and latin indices  $a$  run from 0 to 3, the spatial indexes  $(i, j)$  in  $\{1, 2, 3\}$ . We consider metric of  $\{+, -, -, -, -\}$  signature.

Here  ${}^{(5)}\nabla$  is the covariant derivative compatible with the 5D-metric. 5D-velocities  $\omega^A$  and 4D velocities  $u^A$  are defined respectively as  $\omega^A \equiv dx^A/ds_{(5)}$ ,  $u^A \equiv dx^A/ds$ , with  $\omega^A = \alpha u^A$  and  $g_{AB}\omega^A\omega^B = 1$ ,  $g_{ab}u^au^b = 1$ , where the  $\alpha$  parameter reads:

$$\alpha \equiv \frac{ds}{ds_{(5)}} = \sqrt{g_{ab}\omega^a\omega^b} = \sqrt{1 + \frac{\omega_5^2}{\phi^2}}$$

The dimensional reduction of Eq.3 (see also [9]) provides the set

$$u^a {}^{(4)}\nabla_a u^b = ek \left( \frac{\omega_5}{\sqrt{1 + \frac{\omega_5^2}{\phi^2}}} \right) F^{bc} u_c + \frac{1}{\phi^3} \left( u^b u^c - g^{bc} \right) \partial_b \phi \left( \frac{\omega_5}{\sqrt{1 + \frac{\omega_5^2}{\phi^2}}} \right)^2 \quad (4)$$

$$\frac{d\omega_5}{ds} = 0 \quad (5)$$

where  $F_{ab} = \partial_a A_b - \partial_b A_a$  is the Faraday tensor. Hence a free 5D-test particle becomes a 4D-interacting particle, whose motion is described by (4). Eq.5 provides a constant of motion in agreement with the existence of the Killing vector  $(0, 0, 0, 0, 1)$ . Coupling factors are indeed functions of  $\omega_5$ . In particular the electrodynamics coupling factors, in terms of the effective particle charge-mass ratio  $q/\mu_{(5)}$  is

$$\frac{q}{\mu_{(5)}} = ek \frac{\omega_5}{\sqrt{1 + \frac{\omega_5^2}{\phi^2}}} \quad (6)$$

The right member of the (6) is in general no constant and always upper bounded. Particularly, if we set  $\phi = 1$ , in order to restore the Einstein-Maxwell theory, we have the bound  $q < \mu_{(5)}$  which is unacceptable for every known elementary particle. It could be envisaged how such a problem is related within the background of the geodesic approach to the problem of the huge massive mode of the KK tower ([9, 16, 17]). For  $\omega_5 = 0$ , neutral particle test case, Eq. 6 becomes a geodetic one. Moreover, even in a free-electromagnetic scenario, as prospected in the GSS case, charged ( $\omega_5 \neq 0$ ) particles, being coupled with the extra-dimensional scalar field by a  $\omega_5$ -function do not follow in general a geodetic motion.

### 3.2 Papapetrou approach to motion in KK model

In the geodesic approach to the dynamics, the point-like size of the test particle in  $\mathcal{M}^5$  is assumed. This assumption has been recently discussed in some works [9, 16, 17], where the validity of a model with a point-like particle in a compactified dimension is criticised. A new proposal is given, where, adopting a Papapetrou multipole expansion [18], the particle is described as a localized source in  $M^4$  but still delocalized along the fifth dimension as a consequence of the compactification. Introducing a generic energy-momentum tensor  ${}^{(5)}\mathcal{T}^{AB}$  associated to the body, governed by conservation laws and not depending on the fifth coordinate, like it happens for metric fields, the following equations are considered:

$${}^{(5)}\nabla_A {}^{(5)}\mathcal{T}^{AB} = 0 \quad \partial_5 {}^{(5)}\mathcal{T}^{AB} = 0 \quad (7)$$

Performing a multipole expansion [18] centrad on a trajectory  $X^a$ , at the lowest order the procedure gives the motion equation for a test particle:

$$mu^a {}^{(4)}\nabla_a u^b = (u^b u^c - g^{bc}) \left( \frac{\partial_c \phi}{\phi^3} \right) A + q F^{bc} u_c \quad (8)$$

Below the definitions for coupling factor  $m$ ,  $q$ ,  $A$  and the according definitions for the effective test-particle tensor component follow:

$$m = \frac{1}{u^0} \int d^3x \sqrt{g} \phi^{(5)} \mathcal{T}^{00}, \quad \phi \sqrt{g} T^{\mu\nu} = \int ds m \delta^4(x - X) u^\mu u^\nu \quad (9)$$

$$q = ek \int d^3x \sqrt{g} \phi^{(5)} \mathcal{T}_5^0, \quad ek \phi \sqrt{g} T_5^\mu = \int ds q \delta^4(x - X) u^\mu = \sqrt{g} J^\mu \quad (10)$$

$$A = u^0 \int d^3x \sqrt{g} \phi^{(5)} \mathcal{T}_{55}, \quad \phi \sqrt{g} T_{55} = \int ds A \delta^4(x - X) \quad (11)$$

The parameter  $m$  correctly represents the mass of the particle, which turns out to be localized just in the ordinary 4D space, as it is envisaged by the presence of a 4D Dirac delta function in the above definitions. The equation 8 admits an effective Action which does not coincides to the Action 2. Via an Hamiltonian analysis of such a revised Action, it can be proved that the KK tower of massive modes is suppressed, and the  $q/m$  ratio is no more upper bounded. Indeed, it can be proved that the motion of the particle is correctly governed by a dispersion relation of the form

$$P_\mu P^\mu = m^2, \quad (12)$$

where mass is now variable due to scalar fields. Therefore such an approach allows to deal with test particle consistently without giving up with the compactification hypothesis. Charge  $q$  is still conserved, in consequence of the continuity equation  $\nabla_\mu J^\mu = 0$ , which arises from (7). Mass is in general not conserved and its behaviour is given by

$$\frac{\partial m}{\partial x^\mu} = -\frac{A}{\phi^3} \frac{\partial \phi}{\partial x^\mu}. \quad (13)$$

Therefore the behaviour of mass is related to the variation of the scalar field and the new coupling  $A$  (which has a pure extra-dimensional origin) along the path. An interesting scenario concerning  $A$  to be investigated should be to assume  $A \propto m \phi^2$ : by this way Eq.13 admits an easy integration, providing a power law dependence of mass on the scalar field and , more important, restoring the free falling universality of particle in Eq.8 when a vanishing electromagnetic field is considered.

#### 4 Circular orbits in a Schwarzschild space-time

Let us consider the usual 4D Schwarzschild geometry:

$$ds^2 = \Delta(r) dt^2 - \Delta(r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (14)$$

where  $\Delta(r) = (1 - 2M/r)$ . Using background symmetries, we now restrict to the equatorial geodesics. Tangent vector  $u^\alpha$  to such a curve is  $u^\alpha = dx^\alpha/d\tau = \dot{x}^\mu$ , where we choose  $\tau$  to be the proper time. The metric (14) admits the Killing field  $\xi_t = \partial_t$  and  $\xi_\varphi = \partial_\varphi$ , therefore we have the constants of the motion  $E = g_{\alpha\beta} \xi_t^\alpha u^\beta = \Delta(r)(\dot{t})$  and

$L = -g_{\alpha\beta}\xi_\varphi^\alpha u^\beta = \dot{r}^2 \dot{\varphi}$  that respectively represent the total energy (per unit rest mass) of a particle with respect to a static observer at infinity, and the angular momentum (per unit rest mass) of the particle. Given a particle with rest mass  $\mu$  its dispersion relation  $g_{\alpha\beta}p^\alpha p^\beta = \mu^2$  now reads:

$$E^2 \Delta(r)^{-1} - (\dot{r})^2 \Delta(r)^{-1} - \frac{L^2}{r^2} - \mu^2 = 0 \quad (15)$$

Solving for  $\dot{r}$  we equivalently have  $(\dot{r})^2 = E^2 - \Delta(r) (\mu^2 + L^2/r^2) = E^2 - V^2(r)$ . The effective potential  $V(r)$  is defined by the following formula

$$\frac{V(r)}{\mu} = \sqrt{\Delta(r) \left( 1 + \frac{L^2}{\mu^2 r^2} \right)}$$

and it identifies the value of  $E/\mu$  at which the (radial) kinetic energy of the particle vanishes. Circular orbits correspond to the extrema of the effective potential, therefore solving with respect to  $L$  the equation  $\partial_r V = 0$  we are able to find the angular momentum  $L/\mu$  and then the energy  $E/\mu$  of the particle in a given circular orbit. We have:

$$\frac{E}{\mu} = \frac{(r-2M)}{\sqrt{r(r-3M)}}; \quad \frac{L}{\mu} = \sqrt{\frac{r^2 M}{(r-3M)}}, \quad (16)$$

where for  $r \rightarrow 3M$ ,  $E \rightarrow \infty$ . On the other hand, solving the circular orbits radius  $r/M$  as function of the angular momentum we find the following two solutions

$$\frac{r_\pm}{M} = \frac{L^2 \pm \sqrt{L^2(L^2 - 12M^2\mu^2)}}{2M^2\mu^2} \quad (17)$$

The minimum radius for a stable circular orbit,  $r_{\text{isco}}$ , occurs at the inflection points of the effective potential function, therefore we must solve the equation  $\frac{\partial^2 V}{\partial^2 r} = 0$  (see for example [19], for a generalization to Kerr Newmann metric [20]). In this case  $r_{\text{isco}} = 6M$ : stable circular orbits do not exist at radii smaller than  $r_{\text{isco}}$ . Unstable circular orbits are restricted to the range  $3M < r < 6M$ .

## 5 Generalized Schwarzschild solution

Given the 5D Ricci tensor associated to the 5D metric according to (1), we have the KK equation in vacuum :  ${}^{(5)}R_{AB} = 0$ . The Generalized Schwarzschild Solution (GSS) ([21],[22],[23]) is a stationary, free-electromagnetic solution of 5D-KK equation in vacuum, with 4D-spherical symmetry<sup>3</sup>. After the KK reduction procedure we have equivalently a system of 4D Einstein equation coupled to a massless scalar field:

$$G^{\mu\nu} = \frac{1}{\phi} (\nabla^\mu \partial^\nu \phi), \quad \square \phi = 0. \quad (18)$$

Here  $\nabla^\mu$  is the covariant derivative compatible with the 4D-spacetime metric and  $\square \equiv \nabla^\mu \nabla_\mu$ . We look for a solution of the form

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<sup>3</sup> The ordinary 4D-spacetime  $M^4$  of the direct product  $M^4 \otimes \mathcal{S}^1$  is spherically symmetric; in other words the sections  $t = \text{cost}$ ,  $r = \text{cost}$  and  $x^5 = \text{cost}$  of  $M^5$  are  $S^2$  (spherical surfaces in the ordinary 3D-space).

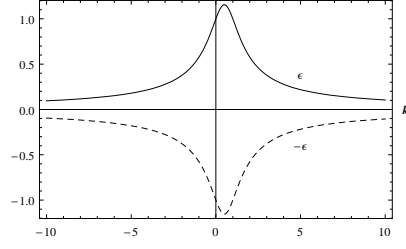


Fig. 1: This figure illustrates the  $\epsilon$ -parameter of the GSS (22), as function of  $k$ -parameter. Solutions  $\epsilon_{\pm}$ , of  $\epsilon^2 (k^2 - k + 1) = 1$ , are plotted. Every points set a metric of the family solutions.

$$ds_{(5)}^2 = A(\rho)dt^2 - B(\rho) \left[ d\rho^2 + \rho^2 d\Omega^2 \right] - C(\rho)dx^5{}^2. \quad (19)$$

Requiring the 3D spherical symmetry and the independence of the metric coefficients from the time  $t$ , we are able to obtain a family of exact solutions of the field equations which are asymptotically flat:

$$ds_{(5)}^2 = \left( \frac{a\rho - 1}{a\rho + 1} \right)^{2\epsilon k} dt^2 - \frac{1}{(a\rho)^2} \frac{(a\rho + 1)^{2[\epsilon(k-1)+1]}}{(a\rho - 1)^{2[\epsilon(k-1)-1]}} \left[ d\rho^2 + \rho^2 d\Omega^2 \right] - \left( \frac{a\rho + 1}{a\rho - 1} \right)^{2\epsilon} dx^5{}^2, \quad (20)$$

The GSS solution is not unique, because in our framework the Birkhoff theorem does not hold [22], and it depends on the real parameter  $(\epsilon, k)$ , which are constant (Fig.1) and constrained by  $\epsilon^2 (k^2 - k + 1) = 1$ . The constant parameter  $a$  is related to the mass of a central body which is supposed to act as source. The Schwarzschild limit is recovered for  $\epsilon \rightarrow 0$ ,  $k \rightarrow \infty$ ; in such a limit  $a = 2c^2/(GM_S)$  -being  $M_S$  the Schwarzschild mass and  $G$  the usual Newton constant- and the above expression turns into the usual 4D exterior solution related to a central body. Noticeably, the Schwarzschild limit is obtained when  $\phi = 1$ . Since we study the exterior solution we are able to perform the following transformation:

$$r = \rho \left( 1 - \frac{rg}{\rho} \right)^2. \quad (21)$$

From Eqs. 20 and 21 a one-parameter family is recovered in the 4D-spherical polar coordinate<sup>4</sup>  $\{t, r, \theta, \varphi\}$  where  $d\Omega^2 \equiv \sin^2 \theta d\varphi^2 + d\theta^2$ . We have:

$$ds_{(5)}^2 = \Delta(r)^{\epsilon k} dt^2 - \Delta(r)^{-\epsilon(k-1)} dr^2 - r^2 \Delta(r)^{1-\epsilon(k-1)} d\Omega^2 - \Delta(r)^{-\epsilon} dx^5{}^2, \quad (22)$$

where  $\Delta(r) = (1 - 2M/r)$ . It is generally used to explore the region  $k \geq 0$  and  $\epsilon \geq 0$  to investigate the physical properties of solutions<sup>5</sup>(22). Within such a range, the GSS solution presents a naked singularity behaviour that resolves<sup>6</sup> in a black hole one only

<sup>4</sup> Consider  $t \in \mathbb{R}$ ,  $r \in ]2M, +\infty[ \subset \mathbb{R}^+$ ,  $\vartheta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi]$

<sup>5</sup> In the cited reference and in [10], for example, is showed how positive density of solution requires  $k > 0$  and for positive mass (as measured at infinity) one must have  $\epsilon k > 0$ .

<sup>6</sup> The event horizon, defined in general coordinates as the surface where the norm of the time-like Killing vector is zero, should be located for the metric family (20) and for  $\epsilon > 0$  and  $k > 0$ , in  $\rho = 1/a$ . Nevertheless the center of the 3-geometry is just at  $\rho = 1/a$ . But in this point the surface area of 2-shells should goes to zero moreover the 5D- Kretschmann scalar and the square of the 4D-Ricci tensor are divergent. Therefore the event horizon shrinks to the singularity in  $\rho = 1/a$ . These kinds of Kaluza Klein solitons are classified as naked singularities

in the Schwarzschild limit for  $(\epsilon, k)$ . Here we consider no negative metric parameters, analysing particle motion in the region  $r > 2M$ , (for a review see [10]). The physical meaning of the metric parameter  $k$ , or alternatively the  $\epsilon$  parameter, has been widely discussed in literature. It is important to note here that this parameter characterises the spacetime external to any astrophysical object described by the selected GSS solution: each values of the  $k$ -constant sets a metric solution as well as the source associated to that solution. Following this interpretation, there should be one different value of the  $k$  parameter associated to one different source. In particular, in a more stringent way in [24,25], the free metric parameter is totally determined by measurements in 4D by taking into account the surface gravitational potential of the astrophysical objects, like the Sun or other stars. Indeed, despite of the naked singularity feature showed far from the Schwarzschild limit, GSS solutions are supposed to describe in principle the exterior spacetimes of any astrophysical sources that satisfy the required metric symmetries and the source is supposed to be embedded in a cloud provided by the scalar field. Generally, to test the validity of such models, many efforts have been made to describe the solar system by a GSS solution; a particular value of  $k$ , adapted to fit the prediction of the standard gravity tests with experimental data, has been associated to the Sun (see for a review [10], see also [11]). Each comparison gives a peculiar estimation for  $k$ ; all these different estimations are based on different tests assumed to probe the model validity. Modelling the Sun by a GSS solution should require a fine tuning of the characteristic parameter. For example, in [26], experimental constraints on equivalence principle violation in the solar system translate in a  $k > 5 \times 10^7$ . Extra dimensions play thus a negligible role in the solar system dynamics. Meanwhile, by measures of the surface gravitational potential in [24,25], the Sun seems to be characterized by a  $k = 2.12$ . On the other hand all the standard tests on light-bending around the Sun, or the perihelion precession of Mercury, constrain  $k \gtrsim 14$ .

## 6 Time-like circular orbits in the GSS spacetimes.

### 6.1 Geodesic approach

Studying the timelike circular orbits in the background (22), at first we consider test particles motion in the geodesic approach where a constant test particle mass  $\mu_{(5)} = \text{const}$  is considered. Let us consider the 5D momentum  ${}^{(5)}P^A \equiv \mu_{(5)}\omega^A = \mu_{(5)}\alpha u^A$  or, equivalently,  ${}^{(5)}P^A \equiv \alpha {}^{(4)}P^A$ , where  ${}^{(4)}P^A \equiv \mu_{(5)}u^A$  and  ${}^{(5)}P^A \equiv \mu_{(5)}\omega^A$ . Granted the three killing vectors

$$\xi_{(t)}^A \equiv \{1, 0, 0, 0, 0\}, \quad \xi_{(5)}^A \equiv \{0, 0, 0, 0, 1\}, \quad \xi_{(\varphi)}^A \equiv \{0, 0, 0, 1, 0\}, \quad (23)$$

the following conserved quantities in  $(M^5, g^5)$  can be defined:

$${}^{(5)}\mathcal{E} \equiv \xi_{(t)}^A {}^{(5)}P_A, \quad {}^{(5)}\Gamma \equiv \xi_{(5)}^A P_A, \quad {}^{(5)}L \equiv \xi_{(\varphi)}^A {}^{(5)}P_A. \quad (24)$$

Introducing the quantities  ${}^{(4)}\mathcal{E} \equiv \xi_{(t)}^A {}^{(4)}P_A$  and  ${}^{(4)}\mathcal{L} \equiv \xi_{(\varphi)}^A {}^{(4)}P_A$ , that are in general non constant along the motion, we can also write:  ${}^{(5)}\mathcal{E} = \alpha {}^{(4)}\mathcal{E}$ ,  ${}^{(5)}L = \alpha {}^{(4)}L$  and  ${}^{(5)}\Gamma = \mu_{(5)}g_{55}\omega^5 = \mu_{(5)}\alpha g_{55}u^5$ . For neutral particles  ${}^{(5)}\mathcal{E} = {}^{(4)}\mathcal{E}$  and  ${}^{(5)}L = {}^{(4)}L$ . The Schwarzschild limit of the constants of motion is

$${}^{(5)}\mathcal{E} = \left(1 - u_5^2\right)^{-1/2} {}^{(4)}\mathcal{E}, \quad {}^{(5)}L = \left(1 - u_5^2\right)^{-1/2} {}^{(4)}L. \quad (25)$$



In this limit  ${}^{(5)}\mathcal{E} = {}^{(4)}\mathcal{E}$  and  ${}^{(5)}L = {}^{(4)}L$  on the surfaces  $x^5 = \text{const}$ , where Eqs.4 are geodesic; hence  ${}^{(4)}\mathcal{E}$  and  ${}^{(4)}L$  are respectively interpreted as the energy at infinity and the total angular momentum. In general, in the case of charged particle, we interpret  ${}^{(5)}\mathcal{E}$  and  ${}^{(5)}L$  as the energy at infinity and the total angular momentum of a particle following the trajectory defined by the first of (4) in  $(M^4, g^4)$ . A 5D-Lagrangian  ${}^{(5)}\mathcal{L}$  and its 4D-counterpart  ${}^{(4)}\mathcal{L}$  are defined as  ${}^{(5)}\mathcal{L} \equiv g_{AB} {}^{(5)}P^A {}^{(5)}P^B$ ,  ${}^{(4)}\mathcal{L} \equiv g_{\mu\nu} {}^{(4)}P^\mu {}^{(4)}P^\nu$ . Explicitly we have:  ${}^{(5)}\mathcal{L} = \alpha^2 {}^{(4)}\mathcal{L} + {}^{(5)}\Gamma^2/g_{55} \equiv \alpha^2 \mu_{(5)}^2 g_{rr}(\dot{r})^2 + {}^{(5)}\mathcal{E}^2/g_{00} + {}^{(5)}L^2/g_{\varphi\varphi} + {}^{(5)}\Gamma^2/g_{55}$  where  $\dot{r} \equiv u^r$ . An effective potential is usually defined via the value of  ${}^{(5)}\mathcal{E}/\mu_{(5)}$  at which the (radial) kinetic energy of the particle vanishes. We have:

$${}^{(5)}V_{eff} \equiv \sqrt{g_{00} \left[ 1 - \left( \frac{{}^{(5)}L^2}{\mu_{(5)}^2 g_{\varphi\varphi}} + \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2 g_{55}} \right) \right]}, \quad \text{or} \quad {}^{(5)}V_{eff} \equiv \sqrt{g_{00} \left( \alpha^2 - \frac{{}^{(5)}L^2}{\mu_{(5)}^2 g_{\varphi\varphi}} \right)}. \quad (26)$$

More explicitly:

$${}^{(5)}V_{eff} \equiv \sqrt{\Delta^{\epsilon k} \left[ 1 + r^2 \Delta^{-1+\epsilon(k-1)} \frac{{}^{(5)}L^2}{\mu_{(5)}^2} + \Delta^\epsilon \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2} \right]} \quad (27)$$

The condition for the occurrence of the circular orbits is:  $\partial {}^{(5)}V_{eff}/\partial r = 0$ . Solving this equation with respect to  ${}^{(5)}L$  we find and the angular momentum  ${}^{(5)}L/(M\mu_{(5)})$  and the energy  ${}^{(5)}\mathcal{E}/\mu_{(5)}$  of a particle in a circular orbit of radius  $r$ :

$$\begin{aligned} {}^{(5)}\mathcal{E} &= \mu_{(5)} \sqrt{\left[ \left( \frac{g_{\varphi\varphi}}{g_{00}} \right)_{,r} \right]^{-1} \left[ g_{\varphi\varphi,r} - \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2} \left( \frac{g_{\varphi\varphi}}{g_{55}} \right)_{,r} \right]} \\ {}^{(5)}L^\pm &= \pm \mu_{(5)} \sqrt{\left[ \left( \frac{g_{00}}{g_{\varphi\varphi}} \right)_{,r} \right]^{-1} \left[ g_{00,r} - \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2} \left( \frac{g_{00}}{g_{55}} \right)_{,r} \right]} \end{aligned}$$

In terms of  ${}^{(4)}\mathcal{E}$  and  ${}^{(4)}L$  we have:

$$\frac{{}^{(5)}\mathcal{E}}{\mu_{(5)}} = \sqrt{1 + \Delta^\epsilon \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2}} \frac{{}^{(4)}\mathcal{E}}{\mu_{(5)}}, \quad \frac{{}^{(5)}L}{\mu_{(5)}M} = \sqrt{1 + \Delta^\epsilon \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2}} \frac{{}^{(4)}L}{\mu_{(5)}M}.$$

The Schwarzschild limit on the energy and angular momentum gives following result:

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{{}^{(5)}L}{\mu_{(5)}M} &= \frac{r}{M} \sqrt{\frac{M}{r-3M} \left( 1 + \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2} \right)} \\ \lim_{k \rightarrow \infty} \frac{{}^{(5)}\mathcal{E}}{\mu_{(5)}} &= \sqrt{\Delta \left( \frac{r-2M}{r-3M} + \frac{{}^{(5)}\Gamma^2}{\mu_{(5)}^2} \frac{2M-r}{3M-r} \right)} \end{aligned}$$

As far as the circular orbit radius  $r_c$  is concerned, we infer, in agreement with the known result in literature:

$$r_c > [1 + \epsilon(2k-1)]M. \quad (28)$$

The above expression is a free- ${}^{(5)}\Gamma$  quantity with  $\lim_{k \rightarrow \infty} r_c = 3M$  and  $r_c < 3M \forall k > 0$ . For an extensive analysis of the motion in the GSS background see [10], [27]-[28]. The above result implies that the last circular orbit does not depend on the particle charge but it is rather a geometrical feature of the selected metric solution. Moreover, particles in circular orbit (stable or unstable) should be detectable also at values  $r < 3M$ .

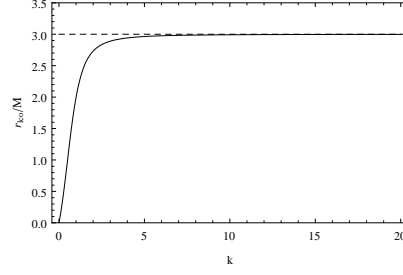


Fig. 2: Test particle last circular orbit radius  $r_{lco} = 1 + (2k - 1)/\sqrt{k^2 - k + 1}$  is plotted as function of the metric parameter  $k$ . The Schwarzschild limit,  $r_{lco} = 3M$ , is also plotted (dashed line). The last circular orbit radius approach  $r_{lco} = 3M$  for large value of  $k$ . Otherwise  $r_{lco} < 3M$ : circular orbits (unstable and stable) are possible also in a region  $r < 3M$ .

## 6.2 Papapetrou analysis

Let us consider now the 4D-dispersion relation (12),  $P_\mu P^\mu = m^2$  where  $P_\mu = mu_\mu$ . Mass is now a varying term, although in this particular case it turns out to be a function of radial coordinate  $r$  only and this means that  $m$  is constant along the circular orbits (at fixed  $r$ ). Only in the Schwarzschild limit, or asymptotically, where  $\phi = 1$  we have  $m = m_0 = \text{const}$ . Anyway, it is always possible to build the constants of motion  $\mathfrak{E}$  and  $\mathfrak{L}$  defined as follows:

$$\mathfrak{E} = p_0 = mg_{00}u^0, \quad \mathfrak{L} = p_\varphi = mg_{\varphi\varphi}u^\varphi. \quad (29)$$

An effective potential for a test particle of mass  $m$  can be defined<sup>7</sup> adopting the standard procedure. We have<sup>8</sup>:

$$\mathfrak{V}_{eff} \equiv \mathfrak{E} = \sqrt{g_{00} \left( m^2 - \frac{\mathfrak{L}^2}{g_{\varphi\varphi}} \right)} \quad (30)$$

We now focus on some interesting scenarios prospected by the Papapetrou approach applied to our analysis of motion into the GSS background. We consider here the case in which the dynamical parameter  $A$  is a function of spacetime point or  $A = \beta m \phi^2$ , where  $\beta$  is a real number. As a particular subcase, imposing  $\beta = 0$  we at first focus on the case  $A = 0$ .

### 6.2.1 $A = 0$

Equations of motion (8) when  $A = 0$  became

$$u^a {}^{(4)}\nabla_a u^b = 0 \quad \text{and} \quad \partial_\mu m = 0. \quad (31)$$

These equations describe a geodetic motion in the ordinary 4D-spacetime for a test particle of constant mass  $m$ , where no scalar field coupling term appears. Formally,

<sup>7</sup> In this case the  $\mathfrak{V}_{eff}$  has unit of mass.

<sup>8</sup> The effective potential now depends of the non constant mass  $m$ , this fact could be alternative seen as a direct dependence of the potential by the matter field  $\phi$ . Anyway we remark that of circular orbits the particle mass turns out to be a constant.

these are the same equations of motions we have in (4), where  $\omega_5 = 0$  and  $\mu_{(5)} = m$ . Nevertheless, Eq.(31) describes charged as well as neutral particles. The following conserved quantities in  $(M^4, g^4)$  can be defined:  $\mathcal{E}_{ek} \equiv \xi_{(t)}^a P_a$ ,  $L_{ek} \equiv \xi_{(\varphi)}^a P_a$ . Equivalently, we have:

$$\mathcal{E}_{ek} = m\Delta^{k\epsilon}\dot{t}, L_{ek} = -m\dot{\varphi}r^2\Delta^{(1-k)\epsilon+1}\csc(\theta)^{-2}. \quad (32)$$

As usually we can define the quantities  $\mathcal{E}$  and  $L$  such that

$$\mathcal{E}_{ek} \equiv \mathcal{E}\Delta^{\epsilon k-1}, \quad L_{ek} \equiv L\Delta^{\epsilon(1-k)+1}. \quad (33)$$

In the Schwarzschild limit  $\mathcal{E}_{ek} = \mathcal{E}$  and  $L_{ek} = L$ ; we interpret the (33) as the energy at infinity and the total angular momentum of the particle. A Lagrangian  $\mathcal{L}_{ek}$  is defined as follows:

$$\mathcal{L}_{ek} \equiv -m^2\Delta^{-\epsilon(k-1)}(\dot{r})^2 + \mathcal{E}_{ek}^2\Delta^{-\epsilon k} - \frac{L_{ek}^2}{r^2}\Delta^{\epsilon(k-1)-1}.$$

The effective potential  $V_{ek} \equiv E/m$  is defined in the usual way as

$$V_{ek} \equiv \sqrt{g_{00}\left(1 - \frac{L_{ek}^2}{m^2 g_{\varphi\varphi}}\right)}.$$

The energy  $\mathcal{E}_{ek}$  and the angular momentum  $L_{ek}$  of a massive test particle in a circular orbit are

$$\mathcal{E}_{ek} = m\sqrt{g_{\varphi\varphi,r}\left[\left(\frac{g_{\varphi\varphi}}{g_{00}}\right)_{,r}\right]^{-1}}, \quad L_{ek}^{\pm} = \pm m\sqrt{g_{00,r}\left[\left(\frac{g_{00}}{g_{\varphi\varphi}}\right)_{,r}\right]^{-1}}, \quad (34)$$

where the Schwarzschild limit provide the following free  $\Gamma_5$ -quantities:

$$\frac{L_{ek}}{Mm} = \frac{r}{M}\sqrt{\frac{1}{\frac{r}{M}-3}}, \quad \frac{\mathcal{E}}{m} = \sqrt{\Delta\frac{r-2M}{r-3M}}.$$

From (34) we infer  $r_c > [1 + \epsilon(2k-1)]M$  for the circular orbits radius  $r_c$  (Cfr.28). The turning points of the effective potential are located in

$$r^{\pm} = \left[1 + \epsilon(3k-2) \pm \epsilon\sqrt{(-1+k)(-1+4k)}\right]M, \quad (35)$$

where last stable circular orbit radius is  $r^+ = r_{\text{isco}}$ , while in the Schwarzschild limit  $r^+ \equiv 6M$  and  $r^- \equiv 2M$ . Moreover it is possible to see, Fig.3, that  $r_{\text{isco}} < 6M$ ,  $\forall k > 0$ . It is worth noting that the last stable circular orbit radius is located below its Schwarzschild limit; this means that in principle there could be particles in stable orbits for values of radius orbit just less than  $6M$ , and this represents a valid constraint in order to compare theory with experimental data. In Tab1 we report the range  $|r_{\text{isco}} - 6M|$ , for selected<sup>9</sup> values of  $k$ . The energy and angular momentum of the last circular orbits are :

$$\frac{L_{ek}^{\pm}}{Mm} = \pm \frac{r_{\text{isco}}^+}{M} \left(1 - \frac{2M}{r_{\text{isco}}^+}\right)^{\frac{1}{2}[(1-k)\epsilon+1]} \sqrt{\frac{\epsilon k}{\epsilon(1-2k) + \left(\frac{r_{\text{isco}}^+}{M} - 1\right)}}, \quad (36)$$

<sup>9</sup> Others constraints of the metric parameters, based on the motion analysis are given in [10], [27], [24], [29], [30], [28].

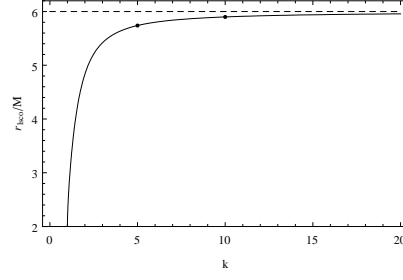


Fig. 3: In this picture last stable circular orbit radius  $r_{\text{lSCO}}/M$ , obtained from the Papapetrou's approach with  $A = 0$  (see (35)), is plotted as function of the  $k$ -parameter. Schwarzschild limit,  $r_{\text{lSCO}} = 6M$ , is also plotted (dashed line). Last stable circular orbit radius approach to the  $6M$  for large value of  $k$ . Otherwise  $r_{\text{lSCO}} < 6M$ : stable circular orbits are possible also in a region  $r < 6M$ .

$$\frac{\mathcal{E}_{\epsilon k}}{m} = \left(1 - \frac{2M}{r_{\text{lSCO}}^+}\right)^{\frac{k\epsilon}{2}} \sqrt{1 - \frac{\epsilon k}{\epsilon(2k-1) + \left(1 - \frac{r_{\text{lSCO}}^+}{M}\right)}}. \quad (37)$$

It is possible to see, Fig.4, that the energy  $\mathcal{E}_{\epsilon k}$  for all values of  $k$ -parameter is always

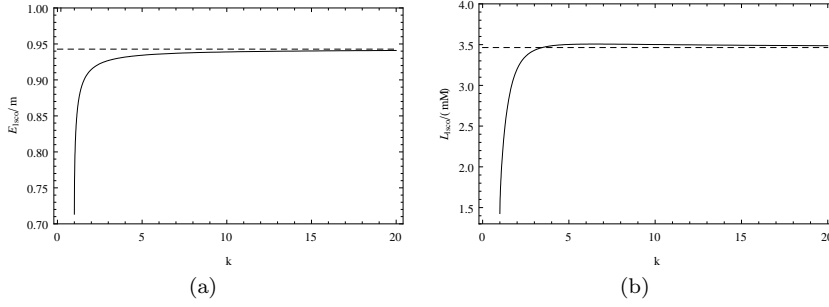


Fig. 4: The “energy”  $E_{\text{lSCO}}/m_0$  and the “angular momentum”  $L_{\text{lSCO}}/m_0$  in the circular orbits, obtained by Papapetrou's approach with  $A = 0$  (see (36) and (37)), are plotted as functions  $k$ -parameter. Schwarzschild' limits for the energy and the angular momentum are also plotted (dashed lines). The energy  $\mathcal{E}_{\text{lSCO}}$  is always below its Schwarzschild limit while  $\mathcal{L}_{\text{lSCO}}$ , for  $k > 3.45644$  is over the Schwarzschild limit.

below its Schwarzschild limit, the angular momentum  $\mathcal{L}_{\epsilon k}$  is beyond the Schwarzschild limit for  $k > 3.45644$ . This fact should not be read as a direct consequence of a possible motion along the fifth dimension, since Eq.(31) does not depend on it, neither on the  $g_{55}$ -metric component. We interpret it as a features related to deformation of the Schwarzschild metric as long as  $k$  is sufficiently small; see also Eq.(34). This seems to be confirmed also by the fact that Eqs.(36, 37) are the same that one can obtain from the geodesic approach with  $\omega_5 = 0$ .

In the following analysis we choice different values of the dynamical parameter  $A$  where  $\mathcal{E}_{\text{lSCO}}$  and  $\mathcal{L}_{\text{lSCO}}$  have the same behaviour.

### 6.3 $A = cost$

As a simplest generalization of the previous case we are going to consider  $A = cost$ . Integrating Eq.(13) along a curve  $\gamma = \gamma(s)$ , between the points  $P = \gamma(s)$  and  $P_0 = \gamma(s_0)$ , we obtain:

$$m = \frac{A}{2\phi^2} + m_0 - \frac{A}{2\phi_0^2}. \quad (38)$$

In the Schwarzschild limit  $m = m_0$ ; we set<sup>10</sup>  $A = 2m_0\phi_0^2$ , therefore  $m = A/2\phi^2$ . Eq.(8) becomes now

$$u^a ({}^4\nabla_a u^b = (u^b u^c - g^{bc}) \left( 2 \frac{\partial_c \phi}{\phi} \right), \quad (39)$$

which does not depend on  $A$ . The effective potential (30) in this case reads:

$$\mathfrak{V}_{eff} = \sqrt{g_{00} \left( \frac{A^2}{4\phi^4} - \frac{\mathfrak{L}^2}{g_{\varphi\varphi}} \right)}. \quad (40)$$

The momentum  $\mathfrak{L}$  and the energy  $\mathfrak{E}$  for circular time-like orbits are :

$$\mathfrak{L}^2 = \frac{A^2}{4\phi^3} \left[ \frac{d}{dr} \left( \frac{g_{00}}{g_{\varphi\varphi}} \right) \right]^{-1} \left[ \frac{d}{dr} \left( \frac{g_{00}}{\phi} \right) - \frac{3g_{00}}{\phi^2} \frac{d\phi}{dr} \right] \quad (41)$$

and

$$\mathfrak{E} = \sqrt{\frac{A^2}{4\phi^3} \left[ \frac{d}{dr} \left( \frac{g_{\varphi\varphi}}{g_{00}} \right) \right]^{-1} \left[ \frac{d}{dr} \left( \frac{g_{\varphi\varphi}}{\phi} \right) - \frac{3g_{\varphi\varphi}}{\phi^2} \frac{d\phi}{dr} \right]}. \quad (42)$$

In the Schwarzschild limit:

$$\mathfrak{L}^2 = \left( \frac{m_0 r \sqrt{M}}{2\sqrt{-3M+r}} \right)^2 \quad \text{and} \quad \mathfrak{E} = \frac{1}{2} \sqrt{-\frac{m_0^2(-2M+r)^2}{(3M-r)r}}. \quad (43)$$

Last circular orbit is located at  $r_{lco} \equiv M[1 + \epsilon(2k-1)]$  - Cf. Eq.(28). For the last stable circular orbit, as the turning point of the effective potential, (40) we have:

$$r_{lsc} \equiv \frac{\sqrt{M^2[4 + (15k-8)\epsilon^2 + 5(8-3k)\epsilon^4] + M[3 + \epsilon(2+k-11\epsilon+5k\epsilon)]}}{(2+k)\epsilon}. \quad (44)$$

We have a free- $A$  quantity, but it is a function of the only metric parameters  $(\epsilon, k)$ . Also in this case  $r_{lsc} < 6M$  and in the Schwarzschild limit  $r_{lsc} = 6M$ . See also Tab1. The energy  $\mathfrak{E}_{lsc}/m_0$  and the momentum  $\mathfrak{L}_{lsc}/(m_0 M)$  in the last stable circular orbits are plotted<sup>11</sup> in Fig.6.

<sup>10</sup> The dynamical parameter  $A$  is here related to the initials conditions of the particle motion.

<sup>11</sup> In the cases  $A = cost$ , and  $A = \beta m \phi^2$  these are not considered as functions of  $m = m(r_{lsc})$  but only  $m_0$ . The comparison with case of  $E_{lsc}/m_{lsc}$  will be detailed discussed in another work [31].

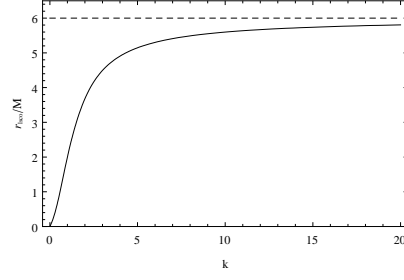


Fig. 5: In this picture last circular orbit radius  $r_{\text{lSCO}}/M$ , obtained from the Papapetrou's approach with  $A = \text{cost}$  (see (44)), is plotted as function of the  $k$ -parameter. Schwarzschild limit,  $r_{\text{lSCO}} = 6M$ , is also plotted (dashed line). Last stable circular orbit radius approach to the  $6M$  for large value of  $k$ . Otherwise  $r_{\text{lSCO}} < 6M$ : stable circular orbits are possible also in a region  $r < 6M$ .

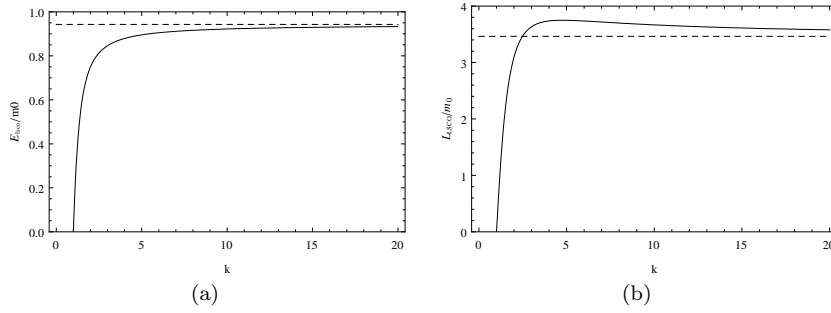


Fig. 6: The energy  $E_{\text{lSCO}}/m_0$  and angular momentum  $L_{\text{lSCO}}/m_0$  of circular orbits, obtained by Papapetrou's approach with  $A = \text{cost}$ , see (42) and (41), are plotted as  $k$ -parameter functions. Schwarzschild limits for the energy and the angular momentum are also plotted (dashed lines). The energy  $\mathcal{E}_{\text{lSCO}}$  is always under its Schwarzschild limit while  $\mathcal{L}_{\text{lSCO}}$ , for  $k > 2.48491$  is over the Schwarzschild limit.

### 6.3.1 $A = \beta m \phi^2$

In the case  $A = \beta m \phi^2$ , where  $\beta$  is a real number, the mass  $m$  is no more a constant, but integrating along a curve  $\gamma = \gamma(s)$ , between the points  $P = \gamma(s)$  and  $P_0 = \gamma(s_0)$ , the following scaling law arises:

$$m = \frac{m_0 \phi_0^\beta}{\phi^\beta}. \quad (45)$$

Here  $m_0 \phi_0^\beta = \text{cost}$  and the equations of motion becomes

$$u^a \text{}^{(4)}\nabla_a u^b = \left( u^b u^c - g^{bc} \right) \frac{\partial_c \phi}{\phi} \beta, \quad (46)$$

therefore it does not depend on  $m$  but on the constant  $\beta$ . The present case reduces to the  $A = \text{cost}$ -case when one sets  $\beta = 2$  and  $A^2 = 4m_0^2 \phi_0^{2\beta}$ . Introducing the parameter  $B^2 \equiv m_0^2 \phi_0^{2\beta}$ , the effective potential reads:

$$\mathfrak{V}_{eff} = \sqrt{g_{00} \left( \frac{B^2}{\phi^{2\beta}} - \frac{\mathfrak{L}^2}{g_{\varphi\varphi}} \right)}. \quad (47)$$

Table 1: Differences  $|r_{lco} - 3M|$  for the last circular orbit radius, and  $|r_{lsc} - 6M|$  for the last stable circular orbit radius are listed for selected values of  $A$ .

$k$	$ r_{lco} - 3M $ ( $M$ )	$A = 0$	$A = cost$	$ r_{lsc} - 6M $ $A = -2m\phi^2$ ( $M$ )	$A = 4m\phi^2$
5	0.04	0.3	0.85	0.411	1.34
10	0.008	0.1	0.40	0.21	0.68
20	0.002	0.04	0.19	0.11	0.34
100	$10^{-4}$	0.008	0.04	0.067	0.22

The constants  $\mathfrak{L}$  and  $\mathfrak{E}$  are respectively given by:

$$\mathfrak{L}^2 = (-1)^\beta \left(1 - \frac{2M}{r}\right)^{(\beta+1-k)\epsilon} \frac{(2M-r)(k+\beta)\epsilon B^2 M r}{M-r+(2k-1)M\epsilon} \quad (48)$$

and

$$\mathfrak{E} = \sqrt{\frac{B^2 \Delta(r)^{(k+\beta)\epsilon} (-1)^\beta [M-r+M(-1+k-\beta)\epsilon]}{M-r+(2k-1)M\epsilon}}. \quad (49)$$

In the Schwarzschild limit they became:

$$\mathfrak{L}^2 = (-1)^\beta \frac{r^2 m_0^2 M}{r-3M}, \quad \mathfrak{E} = \sqrt{-\frac{(-1)^\beta m_0^2 (r-2M)^2}{(3M-r)r}}, \quad (50)$$

where  $B = m_0$  and  $\beta = 2n$  with  $n \in \mathbb{Z}$ . In general, for  $k > -\beta$  last circular orbit is located at  $r_{lco} \equiv M[1 + \epsilon(2k+1)]$  and  $r_{lco} < 3M$ . In the Schwarzschild limit  $r_{lco} = 3M$ . Last stable circular orbit radius, as the turning point of the effective potential (47), is in

$$r_{lsc} \equiv M \frac{3 + \epsilon[k + \beta + (-3 + k + 2k\beta - \beta(2 + \beta))\epsilon]}{(k + \beta)\epsilon} + M \frac{\sqrt{4 + \epsilon^2 [-3k(1 + 2\beta)(\epsilon^2 - 1) + (2 + \beta)(\beta - 4 + (\beta^3 + 2)\epsilon^2)]}}{(k + \beta)\epsilon}$$

while in the Schwarzschild limit  $r_{lsc} = 6M$ . Radius of last stable circular orbit depends on two free parameters,  $k$ , *i.e.* the independent metric parameter and  $\beta$ , namely the “dynamical” one. Moreover  $r_{lsc} < 6M$  for  $\beta > 0$ , while for  $\beta < 0$  and  $k > -\beta$ ,  $r_{lsc} > 6M$  is possible. For  $\beta = 2$  we recover the same physical situations sketched in the case  $A = 0$ .

More generally it is possible to see that at an increase of  $\beta > 0$  for fixed values of the parameter  $k$  provides an increase of the difference  $|r_{lsc} - 6M|$  as listed in Tab.1 for selected values of  $k$  and  $\beta$ .

## 7 Conclusions

The dynamic in Generalized Schwarzschild solution (GSS) spacetimes has been explored studying an effective potential for massive test particles in circular orbits. First we have analysed the motion by the standard approach to the particle dynamics in Kaluza Klein, therefore considering 5D-particles moving along geodesic curves in a

5D-manifold. Then we consider the motion by an approach *à la* Papapetrou to the dynamic considering a 5D particle described by an energy momentum tensor picked along the particle 4D-world-tube. We devoted particular attention to the properties of the four-dimensional counterpart of these solutions in their Schwarzschild limit, in attempts to read the obtained results with the experimental data. A modification of the circular stable orbits has been investigated in agreement with the experimental constraints. In particular, we found in both approaches that, stable circular orbits are possible in general in a region below the Schwarzschild limit ( $r = 6M$ ). We therefore explored the range of possible values of the theory parameters fixing some points in the all range of values that should led to some possible observations.

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